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A MIXTURE THEORY FOR WAVE GUIDE-TYPE PROPAGATION AND DEBONDING IN LAMINATED COMPOSITES

G. A. GURTMAN
Systems, Science and Software, La Jolla, CA 92037, U.S.A.

and

G. A. HEGEMIER
Department of Applied Mechanics and Engineering Science, University of California, San Diego, La Jolla, CA 92037, U.S.A.

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Abstract—A previously developed binary mixture theory for wave guide-type propagation in laminated composites with periodic microstructure is modified to account for debonding. An interface model of the Coulomb frictional-type is postulated and the resulting theory is compared with experimental impact data.

1. INTRODUCTION

In [1] a binary mixture theory for wave guide-type propagation was formulated for both laminated and fiber-reinforced composites with elastic, periodic microstructure. Results using this theory were observed to correlate well with acoustic velocity data for harmonic wave propagation, and transient pulse experiments performed in a shock tube. In both instances, however, the stresses applied to the specimens were of the order of 100 psi, and it was reasonable to assume that the constituents comprising the components were *perfectly* bonded at their interfaces. For more severe impact conditions, it appears likely that the composite might at least partially debond, even in the regime in which the constituents would be expected to retain their structural integrity and indeed still behave elastically. This is the eventuality addressed in the following analysis.

In a recent series of papers by Drumheller[2], Drumheller and Norwood[3], and Drumheller and Lundergan[4], the debonded plate problem was studied both theoretically and experimentally. In [2], it was demonstrated that under dynamic loading, complete debonding with shear-free (lubricated) interfaces leads to an additional degree of freedom within the composite resulting in an additional stable mode of stress wave propagation. Drumheller based his analysis on the exact theory of elasticity and appropriately modified Sve's results for oblique harmonic wave propagation in periodically laminated plates[5]. The additional degree of freedom inherent in the completely debonded laminate analysis leads to a requirement for an additional boundary condition, and [3] addresses this aspect of the problem. The additional boundary condition suggested by Drumheller and Norwood is one in which the warping of the boundary of the debonded laminate is related to the average stresses and displacements in the composite constituents. Reference [4] contains data from a series of experiments which were correlated with the theory in [3] and additional two-dimensional finite difference computer code calculations.

An approach quite different from that of [2–4] will be followed in this paper. In place of the exact theory of elasticity, the mixture theory of [1] will be utilized with essentially a single modification: The interface shear stress boundary condition in [1] is relaxed, and a frictional model of the Coulomb-type is hypothesized in its stead. The resulting system of equations is determinate, and results are seen to correlate well with the experiments.

2. FORMULATION

Following [1]†, consider a periodic array of two linearly elastic, isotropic, homogeneous laminae which are *initially* bonded at their interfaces. A state of plane strain will be assumed in

†For completeness a number of details in [1] concerning the mixture theory construction are reproduced here.

the z-direction, as well as wave motion yielding symmetric u_x and antisymmetric u_y distributions with respect to the y-coordinate within each lamina, where u_x and u_y denote displacements in the x- and y-directions respectively. Consequently, motion of the composite averaged over a typical two layers exists only in the x-direction, and it is sufficient to consider a typical bi-laminate as illustrated in Fig. 1.

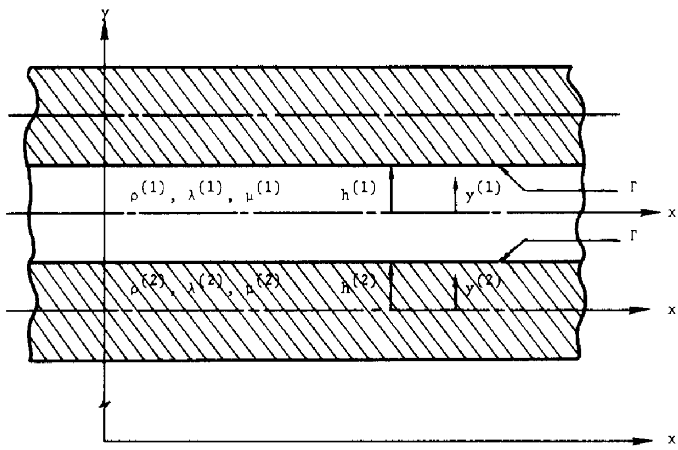

Fig. 1. Laminate geometry and coordinate system.

The basic equations for each lamina are

Equations of motion

$$\partial_x\sigma_{xx}^{(\alpha)} + \partial_y^{(\alpha)}\sigma_{xy}^{(\alpha)} = \rho^{(\alpha)}\partial_t^2 u_x^{(\alpha)}, \tag{2.1a}$$

$$\partial_y^{(\alpha)}\sigma_{yy}^{(\alpha)} + \partial_x\sigma_{xy}^{(\alpha)} = \rho^{(\alpha)}\partial_t^2 u_y^{(\alpha)}, \tag{2.1b}$$

Constitutive relations

$$\sigma_{xx}^{(\alpha)} = (\lambda + 2\mu)^{(\alpha)}\partial_x u_x^{(\alpha)} + \lambda^{(\alpha)}\partial_y^{(\alpha)}u_y^{(\alpha)}, \tag{2.2a}$$

$$\sigma_{yy}^{(\alpha)} = (\lambda + 2\mu)^{(\alpha)}\partial_y^{(\alpha)}u_y^{(\alpha)} + \lambda^{(\alpha)}\partial_x u_x^{(\alpha)}, \tag{2.2b}$$

$$\sigma_{xy}^{(\alpha)} = \mu^{(\alpha)}(\partial_y^{(\alpha)}u_x^{(\alpha)} + \partial_x u_y^{(\alpha)}), \tag{2.2c}$$

where

$$\partial_x(\,) \equiv \partial(\,)/\partial_x,\ \partial_y^{(\alpha)}(\,) \equiv \partial(\,)/\partial y^{(\alpha)},\ \partial_t^n(\,) \equiv \partial^n(\,)/\partial_t^n;$$

σ_{xx}, σ_{xy}, σ_{yy} are components of the stress tensor; $\rho^{(\alpha)}$, $\lambda^{(\alpha)}$, $\mu^{(\alpha)}$ denote, respectively, mass density and Lamé constants of material α; t represents time; $y^{(\alpha)}$ is the local coordinate in the y-direction with origin at the mid-plane of the α-constituent; and the superscript $\alpha = 1, 2$ refers to the α-constituent.

In addition to equations (2.1) and (2.2), the complete specification of motion requires:

Symmetry

$$u_y^{(\alpha)} = 0,\ \sigma_{xy}^{(\alpha)} = 0 \text{ on } y^{(\alpha)} = 0. \tag{2.3}$$

Interface conditions. Consider next the interface Γ defined by $y^{(1)} = h^{(1)}$, $y^{(2)} = -h^{(2)}$ in the initial configuration. For purposes of this analysis, a Coulomb–Mohr bond *failure* criteria on Γ will be postulated in the form

$$|\sigma^*_{xy_{cr}}| = A - \kappa\sigma^*_{yy} \tag{2.4}$$

where the star refers to Γ, and A, κ are real non-negative constants. The coefficient of cohesion, A, accounts for any shear strength of the bond which may be present when the normal stress acting at the interface is tensile rather than compressive; the quantity κ represents a coefficient of friction. Figure 2 illustrates allowable bond states in stress space. Only those states within or on

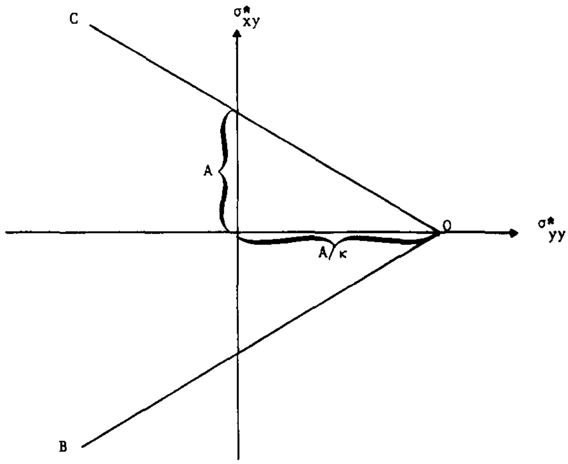

Fig. 2. Failure surface in stress space.

the wedge BOC may exist. Should the tensile stress σ^*_{yy} exceed a value A/κ, the problem is no longer well posed.† Within the wedge BOC, the bond is *perfect* and the following interface conditions shall be imposed:

$$\partial_t u_x^{(1)} = \partial_t u_x^{(2)},\; u_y^{(1)} = u_y^{(2)},\; \sigma_{xy}^{(1)} = \sigma_{xy}^{(2)},\; \dot{\sigma}_{yy}^{(1)} = \sigma_{yy}^{(2)}. \tag{2.5}$$

On OC or OB the interface has *debonded*; in this case the interface conditions are assumed in the form:

$$u_y^{(1)} = u_y^{(2)},\; \sigma_{yy}^{(1)} = \sigma_{yy}^{(2)} \equiv \sigma^*_{yy}, \tag{2.6a}$$

$$\sigma_{xy}^{(1)} = \sigma_{xy}^{(2)} \equiv \sigma^*_{xy} = (A - \kappa\sigma^*_{yy})\,\mathrm{sgn}\,(\partial_t u_x^{(1)} - \partial_t u_x^{(2)}). \tag{2.6b}$$

The relations (2.6) permit relative translation in the x-direction of the laminate constituents along Γ.

If, during the course of a particular problem, the interfacial shear stress causes bond breakage and then, at some later time, drops within the wedge BOC, the laminae are assumed to *rebond* and conditions (2.5) are again invoked. The way in which the debonded model is being used, particularly the inclusion of the A term, implies that a "healing process" takes place at the interface. It has been suggested by one of the paper's reviewers that provided the broken bonds were not exposed to oxygen, are in the fractured state for times on the order of μ seconds, and undergo relatively small displacements, certain polymers, such as PMMA, may be capable of rebonding. The authors sought confirmation of this point, but were unable to obtain definitive experimental verification in the literature.

Initial and boundary data. Initial conditions at $t = 0$, and appropriate boundary data on $x = 0, L$ where $x\epsilon(0, L)$, $L \leq \infty$, are necessary to complete the formulation.

3. ANALYSIS

Mixture equations of motion

If (2.1a) is integrated with respect to $y^{(\alpha)}$ from 0 to $h^{(\alpha)}$, and averaged stresses and displacements are defined according to

†It is recognized that the conditions of continuity of normal stresses and velocities in the case of delamination will only be satisfied for the case of relatively long compressive pulses. Instances in which voids form at laminae interfaces will be treated in a forthcoming paper.

$$(\ )^{(\alpha a)} \equiv \frac{1}{h^{(\alpha)}} \int_0^{h^{(\alpha)}} (\ )^{(\alpha)} \, dy^{(\alpha)} \tag{3.1}$$

where $h^{(\alpha)}$ denotes the half thickness of laminae of material α, one obtains

$$h^{(\alpha)} \partial_x \sigma_{xx}^{(\alpha a)} - h^{(\alpha)} \rho^{(\alpha)} \partial_t^2 u_x^{(\alpha a)} = -\sigma_{xy}^{(\alpha)}(x, h^{(\alpha)}, t). \tag{3.2}$$

Since $\sigma_{xy}^{(\alpha)}$ must be continuous across laminae interfaces in either the bonded or delaminated cases, and is antisymmetric in $y^{(\alpha)}$, one notes that

$$-\sigma_{xy}^{(1)}(x, h^{(1)}, t) = \sigma_{xy}^{(2)}(x, h^{(2)}, t) \equiv \sigma_{xy}^{*}(x, t). \tag{3.3}$$

With the aid of (3.3), equations (3.2) can be placed in a binary mixture form following the introduction of "partial" stresses and densities:

$$\sigma_{xx}^{(\alpha p)} \equiv n^{(\alpha)} \sigma_{xx}^{(\alpha a)}, \ \rho^{(\alpha p)} \equiv n^{(\alpha)} \rho^{(\alpha)} \tag{3.4}$$

where

$$n^{(\alpha)} \equiv h^{(\alpha)} / (h^{(1)} + h^{(2)}) \tag{3.5}$$

is the volume fraction of the α-constituent. Utilizing (3.3) and (3.4), the momentum equations (3.2) become

$$\partial_x \sigma_{xx}^{(1p)} - \rho^{(1p)} \partial_t^2 u_x^{(1a)} = P \tag{3.6a}$$

$$\partial_x \sigma_{xx}^{(2p)} - \rho^{(2p)} \partial_t^2 u_x^{(2a)} = -P \tag{3.6b}$$

where

$$P = \sigma_{xy}^{*} / (h^{(1)} + h^{(2)}) \tag{3.7}$$

is an "interaction term" reflecting momentum transfer from one constituent to another via shear interaction at laminate interfaces.

Mixture constitutive relations

If (2.2a) and (2.2b) are averaged according to (3.1) and continuity of $u_y^{(\alpha)}$ across laminate interfaces is invoked, one obtains

$$n_1 \left(\frac{\sigma_{xx}^{(1a)}}{\lambda^{(1)}} - \frac{E^{(1)}}{\lambda^{(1)}} \partial_x u_x^{(1a)} \right) = -S, \tag{3.8a}$$

$$n_2 \left(\frac{\sigma_{xx}^{(2a)}}{\lambda^{(2)}} - \frac{E^{(2)}}{\lambda^{(2)}} \partial_x u_x^{(2a)} \right) = S, \tag{3.8b}$$

$$n_1 \left(\frac{\sigma_{yy}^{(1a)}}{E^{(1)}} - \frac{\lambda^{(1)}}{E^{(1)}} \partial_x u_x^{(1a)} \right) = -S, \tag{3.8c}$$

$$n_2 \left(\frac{\sigma_{yy}^{(2a)}}{E^{(2)}} - \frac{\lambda^{(2)}}{E^{(2)}} \partial_x u_x^{(2a)} \right) = S, \tag{3.8d}$$

where

$$S = u_y^{*}(x, t) / (h^{(1)} + h^{(2)}), \tag{3.9a}$$

$$u_y^{*} \equiv u_y^{(2)}(x, h^{(2)}, t) = u_y^{(1)}(x, -h^{(1)}, t) = -u_y^{(1)}(x, h^{(1)}, t), \tag{3.9b}$$

$$E^{(\alpha)} \equiv (\lambda + 2\mu)^{(\alpha)}. \tag{3.9c}$$

Subject to the constraint mentioned in footnote†, equations (3.8) and (3.9) are valid for both bonded and delaminated laminates. The quantity S represents a constitutive "interaction" term.

Construction of interaction terms

In order to place (3.6) and (3.8) in a determinate binary mixture form, it is necessary to derive expressions for the interaction terms P and S in terms of the dependent variables $u_x^{(1a)}$, $u_x^{(2a)}$ of the mixture momentum equations (3.6). To facilitate this task, the construction procedure outlined in [1] may be employed. The procedure commences by assuming a power-series expansion for stresses and displacements about the mid-plane of each laminae. If one denotes stresses or displacements in the α-constituent by $g^{(\alpha)}$, then

$$g^{(\alpha)}(x, y^{(\alpha)}, t) = g^{(\alpha)}_{(0)}(x, t) + g^{(\alpha)}_{(1)}(x, t) + \cdots + g^{(\alpha)}_{(n)}(x, t) y^{(\alpha)^n}/n! + \cdots \tag{3.10}$$

It is noted that the series (3.10) need not be convergent, but only asymptotic in a parameter ϵ as $\epsilon \to 0$, where ϵ represents the ratio of typical micro-to-macro-dimensions of the composite.

Substituting (3.10) into (2.1) and (2.2) and equating terms of similar order of $y^{(\alpha)}$, one obtains differential-recurrence formulae for the coefficients $g^{(\alpha)}_{(n)}$:

$$\partial_x \sigma^{(\alpha)}_{xx(n)} + \sigma^{(\alpha)}_{xy(n+1)} = \rho^{(\alpha)} \partial_t^2 u^{(\alpha)}_{x(n)}, \tag{3.11a}$$

$$\sigma^{(\alpha)}_{yy(n+1)} + \partial_x \sigma^{(\alpha)}_{xy(n)} = \rho^{(\alpha)} \partial_t^2 u^{(\alpha)}_{y(n)}, \tag{3.11b}$$

$$\sigma^{(\alpha)}_{xx(n)} = E^{(\alpha)} \partial_x u^{(\alpha)}_{x(n)} + \lambda^{(\alpha)} u^{(\alpha)}_{y(n+1)}, \tag{3.11c}$$

$$\sigma^{(\alpha)}_{yy(n)} = E^{(\alpha)} u^{(\alpha)}_{y(n+1)} + \lambda^{(\alpha)} \partial_x u^{(\alpha)}_{x(n)}, \tag{3.11d}$$

$$\sigma^{(\alpha)}_{xy(n)} = \mu^{(\alpha)} (u^{(\alpha)}_{x(n+1)} + \partial_x u^{(\alpha)}_{y(n)}). \tag{3.11e}$$

Equations (3.11a), (3.11c,d) apply for $n = 0, 2, 3, \ldots$, whereas (3.11b), (3.11e) are valid for $n = 1, 3, 4, \ldots$. It is noted that

$$\sigma^{(\alpha)}_{xy(n)} = u^{(\alpha)}_{y(n)} = 0 \quad \text{for} \quad n = 0, 2, 4, \ldots, \tag{3.11f}$$

$$\sigma^{(\alpha)}_{xx(n)} = \sigma^{(\alpha)}_{yy(n)} = u^{(\alpha)}_{x(n)} = 0 \quad \text{for} \quad n = 1, 3, 5, \ldots. \tag{3.11g}$$

Conditions (3.11f,g) follow from symmetry or asymmetry of appropriate stresses and displacements.

Using equations (3.11), the series (3.10) can be telescoped and written

$$g^{(\alpha)}(x, y^{(\alpha)}, t) = \Big(\sum_n \mathscr{L}^{(\alpha)}_{(n)}(x, t) y^{(\alpha)^n}/n!\Big) u_x^{(1a)} + \Big(\sum_n \mathscr{M}^{(\alpha)}_{(n)}(x, t) y^{(\alpha)^n}/n!\Big) \sigma^{(\alpha p)}_{xx} \tag{3.12}$$

where $\mathscr{L}^{(\alpha)}_{(n)}$ and $\mathscr{M}^{(\alpha)}_{(n)}$ are linear differential operators with respect to x and t.

As in [1], the objective here is the development of a *first order* mixture theory. Consequently it will be necessary to derive only the first term or two of the foregoing operators. Details concerning the complete determination of all operators (and resulting higher order theories for the bonded case) can be found in [6–8].

Now, let the characteristic dominant signal wavelength of wave motion be l, and the typical composite microdimension be $h^{(1)} + h^{(2)}$. Consider the nondimensional variables

$$\xi = x/l, \; \zeta^{(\alpha)} = y^{(\alpha)}/(h^{(1)} + h^{(2)}), \tag{3.13}$$
$$\tau = tc_0/l, \; \epsilon = (h^{(1)} + h^{(2)})/l,$$

where c_0 denotes a representative "mixture" velocity. Then, if $x \epsilon (0, l)$, $t \epsilon (0, l/c_0)$, we have

$\xi\epsilon(0,1)$, $\tau\epsilon(0,1)$; i.e. the typical macrodimension is now $O(1)$ whereas the typical microdimension is $O(\epsilon)$.

With the aid of the recurrence relations (3.11), the nondimensional variables (3.13), and the premise that l may be selected such that $\partial_\xi(\,)=O(1)$, $\partial_\tau(\,)=O(1)$, where ( ) represents $u_x^{(\alpha a)}$ or $\sigma_{xx}^{(\alpha a)}$, the stress continuity condition $\sigma_{yy}^{(1)}(x,h^{(1)},t)=\sigma_{yy}^{(2)}(x,-h^{(2)},t)$ can be written

$$[1+O(\epsilon^2)]\sigma_{yy}^{(1a)}=[1+O(\epsilon^2)]\sigma_{yy}^{(2a)}. \tag{3.14}$$

As in [1], it will be assumed here that $\epsilon^2 \ll 1$. Consequently the approximation

$$\sigma_{yy}^{(1a)}\simeq\sigma_{yy}^{(2a)} \tag{3.15}$$

is adopted, whereupon (3.8c,d) yield

$$S=(\lambda^{(1)}\partial_x u^{(1a)}-\lambda^{(2)}\partial_x u^{(2a)})/E \tag{3.16a}$$

where

$$E\equiv\frac{E(1)}{n^{(1)}}+\frac{E(2)}{n^{(2)}}. \tag{3.16b}$$

Substitution of (3.16) into (3.8a,b) furnishes

$$\sigma_{xx}^{(1p)}\simeq C_{11}\partial_x u_x^{(1a)}+C_{12}\partial_x u_x^{(2a)} \tag{3.17a}$$

$$\sigma_{xx}^{(2p)}\simeq C_{12}\partial_x u_x^{(1a)}+C_{22}\partial_x u_x^{(2a)} \tag{3.17b}$$

where

$$C_{\alpha\alpha}=n^{(\alpha)}E^{(\alpha)}-\frac{\lambda^{(\alpha)2}}{E},\quad C_{\alpha\beta}=\frac{\lambda^{(\alpha)}\lambda^{(\beta)}}{E},\quad (\alpha,\beta=1,2;\ \alpha\neq\beta). \tag{3.17c}$$

Equations (3.17) constitute first order mixture-constitutive equations.

Finally, to obtain a first order expression for P in the case of a *bonded* interface, multiply (2.1b) by $y^{(\alpha)}$ and integrate as follows:

$$\frac{1}{h^{(\alpha)}}\int_0^{h^{(\alpha)}} y^{(\alpha)}\left[\partial_y^{(\alpha)}u_x^{(\alpha)}+\partial_x u_y^{(\alpha)}-\frac{\sigma_{xy}^{(\alpha)}}{\mu^{(\alpha)}}\right]\,\mathrm{d}y=0. \tag{3.18}$$

Upon expanding $(\partial_x u_y^{(\alpha)}-\sigma_{xy}^{(\alpha)})/\mu^{(\alpha)}$ in powers of $y^{(\alpha)}$ according to (3.10), and subsequently integrating by parts, one finds

$$u_x^{(\alpha)}(x,h^{(\alpha)},t)-u_x^{(\alpha a)}+\frac{h^{(\alpha)2}}{3}\left(\partial_x u_{y(1)}^{(\alpha)}-\frac{1}{\mu^{(\alpha)}}\sigma_{xy(1)}^{(\alpha)}\right)+\cdots=0. \tag{3.19}$$

With use of the recurrence relations (3.11), the higher order terms in (3.19) may be grouped in the form

$$u_x^{(\alpha)}(x,h^{(\alpha)},t)-u_x^{(\alpha a)}+\frac{h^{(\alpha)}}{3}\Big[(1+\eta_1^{(\alpha)}\epsilon^2+\cdots)\partial_x u_y^{(\alpha)}(x,h^{(\alpha)},t)$$
$$-\frac{1}{\mu^{(\alpha)}}(1+\eta_2^{(\alpha)}\epsilon^2+\cdots)\sigma_{xy}^{(\alpha)}(x,h^{(\alpha)},t)\Big]=0 \tag{3.20}$$

where $\eta_\beta^{(\alpha)}$ are second order differential operators in ξ and τ. As before, since $\epsilon^2\ll 1$, all $O(\epsilon^2)$ terms in (3.20) shall be neglected. Under this approximation, and following use of (3.9b) and (3.3), one finds

$$u_x^{(1)}(x,h^{(1)},t)-u_x^{(1a)}+\frac{h^{(1)}}{3}\left(\frac{1}{\mu^{(1)}}\sigma_{xy}^{*}-\partial_x u_y^{*}\right)=0,$$

$$\ldots^{(2)}, t) - u_x^{(2a)} + \frac{h^{(2)}}{3}\left(\partial_x u_y^* - \frac{1}{\mu^{(2)}}\sigma_{xy}^*\right) = 0. \tag{3.21}$$

If one no[w] …vative of equations (3.21) with respect to t, and applies the interface cond[itions]

$$\begin{aligned}\partial_t u_x^{(1)}(x, h^{(1)}, t) &= \partial_t u_x^{(1)}(x, -h^{(1)}, t)\\ &= \partial_t u_x^{(2)}(x, h^{(2)}, t),\end{aligned} \tag{3.22}$$

expression can be solved for σ_{xy}^* as follows:

$$\partial_t \sigma_{xy}^* = \frac{3\mu^{(1)}\mu^{(2)}}{h^{(1)}\mu^{(2)} + h^{(2)}\mu^{(1)}}\left\{\left[1 + \frac{\lambda_1 \epsilon^2}{3E}\partial_\xi^2\right]\partial_t u_x^{(1a)} - \left[1 + \frac{\lambda_2 \epsilon^2 \partial_\xi^2}{3E}\right]\partial_t u_x^{(2a)}\right\}. \tag{3.23}$$

If $\epsilon^2 \ll 1$, then $O(\epsilon^2)$ terms in (3.23) can be neglected. Under this approximation, the interaction term $P = \sigma_{xy}^*/(h^{(1)} + h^{(2)})$ takes the form

$$\partial_t P \simeq \frac{3\mu^{(1)}\mu^{(2)}}{(n^{(1)}\mu^{(2)} + n^{(2)}\mu^{(1)})(h^{(1)} + h^{(2)})^2}(\partial_t u_x^{(1a)} - \partial_t u_x^{(2a)}) \tag{3.24}$$

if $|\sigma_{xy}^*| < |\sigma_{xyCR}^*|$.

If *debonding* has occurred, then equations (2.6b), (3.7), (3.21) lead immediately to the following expression for the interaction term P:

$$P \simeq \frac{(A - \kappa\sigma_{yy}^*)}{(h^{(1)} + h^{(2)})}\,\mathrm{sgn}\,(\partial_t u_x^{(1a)} - \partial_t u_x^{(2a)}) \tag{3.25a}$$

if $|\sigma_{xy}^*| = |\sigma_{xyCR}^*|$.

Here $\sigma_{yy}^* \equiv \sigma_{yy}^{(1)}(x, h^{(1)}, t) = \sigma_{yy}^{(2)}(x, -h^{(2)}, t)$. But, using (3.12), $\sigma_{yy}^{(\alpha)}$ and $\sigma_{yy}^{(\alpha a)}$ can be shown to differ by $O(\epsilon^2)$. To first order, therefore, (3.15) and (3.8) yield

$$\sigma_{yy}^* \simeq \frac{n^{(1)}\lambda^{(1)}E^{(2)}}{n^{(1)}E^{(2)} + n^{(2)}E^{(1)}}\partial_x u_x^{(1a)} + \frac{n^{(2)}\lambda^{(2)}E^{(1)}}{n^{(1)}E^{(2)} + n^{(2)}E^{(1)}}\partial_x u_x^{(2a)}. \tag{3.25b}$$

4. COMPARISON OF NUMERICAL AND EXPERIMENTAL RESULTS

The momentum equations (3.6), interaction relations (3.25a–3.25b), failure criteria (2.4), and constitutive equations (3.17) define a complete set of mixture equations for wave propagation in both bonded and delaminated composites. Unlike the analysis of a perfectly lubricated delaminated wave guide presented in [3], no additional boundary conditions need be specified. This is due to the fact that in the case in which friction forces give rise to momentum coupling between laminae, warping of the boundary remains a localized phenomena and is not transmitted throughout the interior of the delaminated composite. Specification of either stresses, displacements, or velocities in each laminate at $x = 0, L$ therefore, supplies the necessary boundary conditions. In the following, calculations with $L = \infty$, quiescent initial data, and velocity boundary conditions on $x = 0$ are described.

A series of experiments dealing with delaminated plates were reported by Drumheller and Lundergan in [4]. Their composites were composed of alternating layers of Polymethyl methacrylate (PMMA, Rohm and Haas Type A) 0·762 mm thick, and 6061-T6 aluminum 0·792 mm thick. The laminae of the composite were oriented perpendicular to the impact plane, and struck by a projectile fired from a 10 cm bore light gas gun. An aluminum buffer plate 1·0 cm thick was placed at the rear of the composite to improve planarity of the transmitted wave front. A transparent Dynasil 100 window followed the buffer, and a thin mirror was vapor deposited at the buffer window interface. The motion of this mirror was monitored to within $\pm 1{\cdot}5 \times 10^{-5}$ mm by means of a displacement interferometer. The experimental configuration, reproduced from [4], is depicted in Fig. 3. Table 1 summarized the shot matrix.

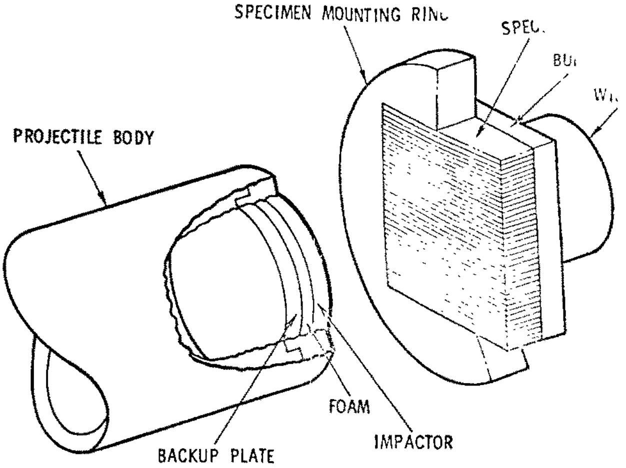

Fig. 3. Experimental arrangement for Sandia experiments (reproduced from [4]).

Table 1. The experimental configurations*

Experiment No.	Composite Thickness (cm)	Projectile		
		Material	Velocity (cm/μsec)	Thickness (cm)
1†	0.820	Aluminum	0.001355	1.634
2	0.776	Aluminum	0.001420	1.631
3	0.806	Aluminum	0.001144	1.571
4	0.803	PMMA	0.002118	0.691
5	0.805	PMMA	0.003078	0.694
6	0.812	Tungsten Carbide	0.001289	0.975
7	0.810	Aluminum	0.001330	0.656
8	0.809	Aluminum	0.001066	0.164

*In all experiments the buffer plate was 1 cm thick. The bond thicknesses were all less than 0.0002 cm thick.

†Experiment 1 exhibited excessive flyer tilt and Drumheller and Lundergan considered the data useless.

The analysis described in Section 2 was coded in finite difference form and solved using a UNIVAC 1108 electronic computer. Material properties for the composite laminates, flyer plate material, buffer and window materials are listed in Table 1. Input to the computer code required only the material properties of all constituents, geometries of the test configuration, and flyer impact velocity. Numerical values for A and κ were determined by a parametric study on the results of experimental shot number 2. They were found to be $A = 0{\cdot}01 \times 10^9$ dynes/cm^2 and $\kappa = 0{\cdot}50$. All other correlations were performed using these values. Results are depicted in Figs. 4–10. Absolute arrival times were not measured during the gas gun shots, and the origin of time scales on the experimental traces corresponded to initial motion of the aluminum Dynasil 100 interface. These traces were overlaid on the calculated results more or less at first peak arrival. The absolute times in all figures correspond to those predicted by the theory. As can be seen, agreement is generally excellent. Both theoretical and experimental results clearly show the effects of bond breakage and delamination. This manifests itself through the appearance of a precursor at the front of the wave profile (see Figs. 4–9), and greatly reduce oscillations behind the wavefront when compared with both theoretical and experimental results for rigidly bonded waveguides (see [1]).

While no experimental determination of interfacial shear stress was made directly in [4], it was possible to monitor this quantity in the code calculations. Figure 11 illustrates the behavior of σ^*_{xy} as a function of time for experimental condition 1 at a location 0·406 cm behind the

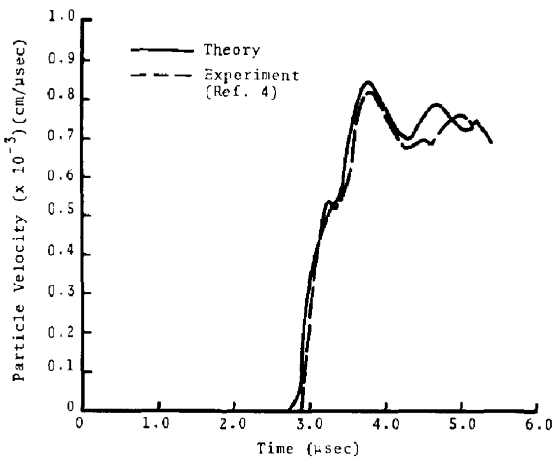

Fig. 4. Comparison of theory with experiment No. 2.

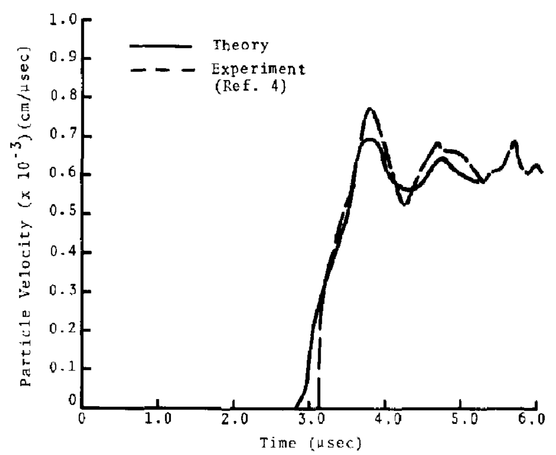

Fig. 5. Comparison of theory with experiment No. 3.

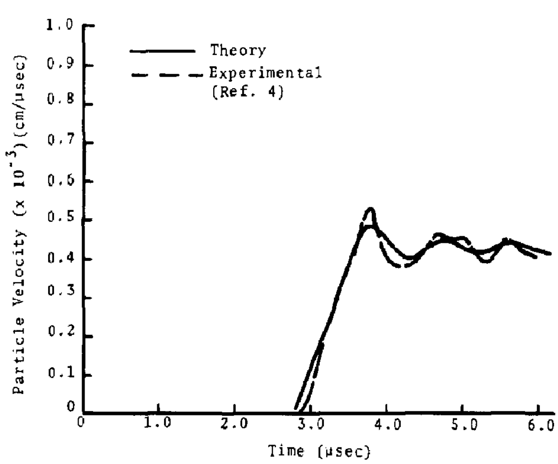

Fig. 6. Comparison of theory with experiment No. 4.

composite's front face. An identical calculation was performed assuming perfect bonding, and is included in Fig. 11 for purposes of comparison. According to the calculations, the interface underwent slippage (i.e. shear stress was on the failure surface COB) from 0·693 μsec to 1·14 μsec following initial flyer impact. Before and after these times, the shear stress fell within the failure surface, and the bond was rigid. The average velocity profiles at $x = 0{\cdot}406$ cm for

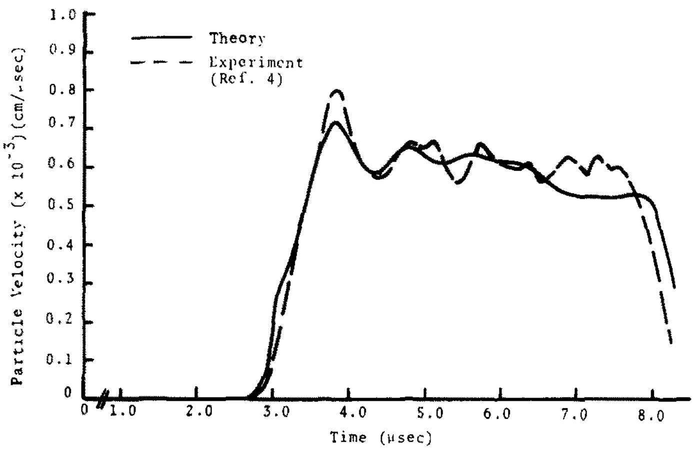

Fig. 7. Comparison of theory with experiment No. 5.

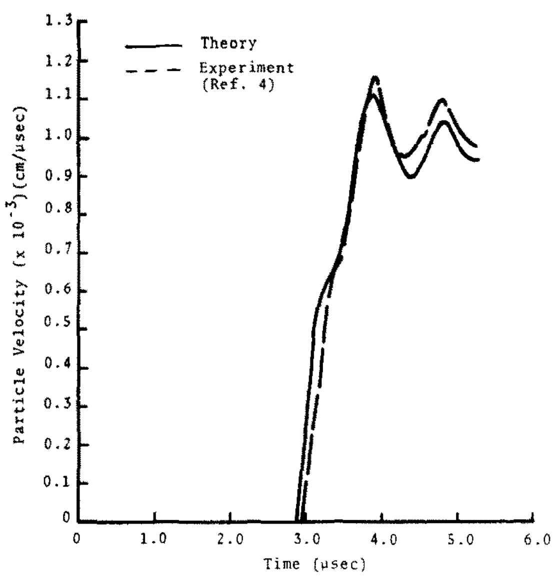

Fig. 8. Comparison of theory with experiment No. 6.

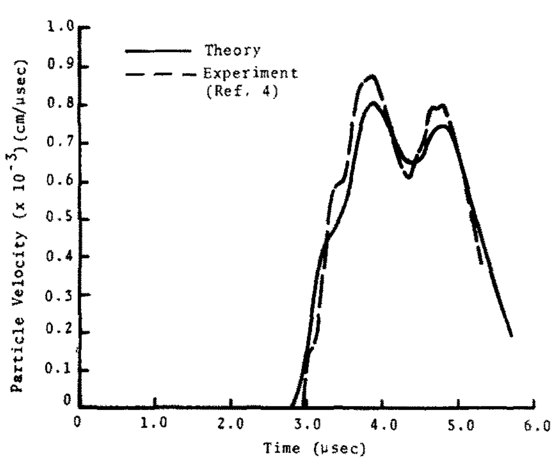

Fig. 9. Comparison of theory with experiment No. 7.

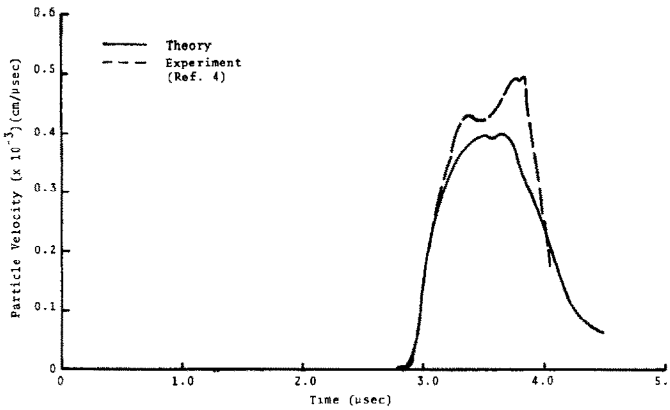

Fig. 10. Comparison of theory with experiment No. 8.

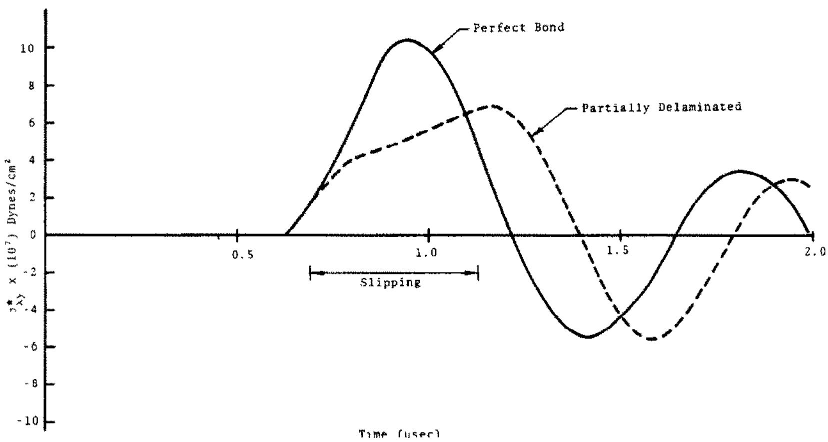

Fig. 11. Calculation of interfacial shear stress for experiment No. 1 ($x = 0{\cdot}406$ cm).

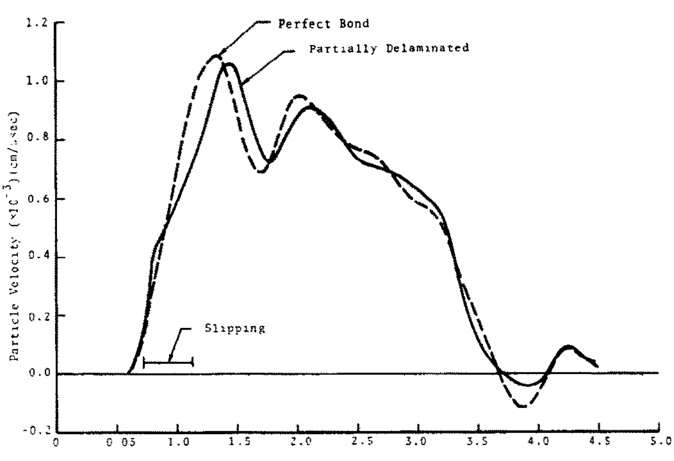

Fig. 12. Calculation of average particle velocity for experiment No. 1 ($x = 0{\cdot}406$ cm).

experimental condition 1 are depicted in Fig. 12 for both the rigid bond and partially delaminated cases. These indicate a lower peak velocity when the laminate experiences bond breakage and a reduced frequency of oscillation following passage of the wavefront.

The failure and slip model hypothesized in this paper appears plausible from a physical standpoint, and while quite elementary, yields excellent correlation with the experimental results. Clearly however, it is only one of many such models which might be devised. Only direct measurements will suffice to totally resolve the question as to the correct formulation of the interlaminar shear stresses.

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